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Spectra of *p*-adic Schrödinger-type operators with random radial potentials

Denis Krutikov

Universität Essen, Fachbereich Mathematik/Informatik, 45117 Essen, Germany

E-mail: denis.krutikov@uni-essen.de

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Abstract
We study spectra of p-adic Schrödinger-type operators with random radial potentials in two different models. Spectral properties of nonrandom 2-adic Schrödinger-type operators are also investigated.

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1. Introduction

The initial impetus to the development of the p-adic mathematical physics, and in particular of the p-adic quantum theory, was given by the hypothesis about a possible non-Archimedean structure of space–time at sub-Planck distances, which takes origin in the inequality

$$\Delta x \geqslant l_{pl} = \sqrt{\frac{hG}{c^3}}.$$

Here x is an uncertainty in a length measurement, h is the Planck constant, c is the velocity of light and G is the gravitational constant. This inequality was proved in quantum gravity (string theory). It implies that at extremely small distances smaller than the Planck length l_{pl} (approximately 10^{-35} m) a measurement of distances is impossible and therefore the so-called Archimedean axiom of measurement does not hold. So we have to replace the usual Archimedean geometry, described by means of real numbers, by the non-Archimedean one, if we want to describe processes at sub-Planck distances. The convenient number fields are the p-adic number fields $\mathbb{Q}_p$ (p is some prime integer), because of the non-Archimedean triangle inequality $|x + y|_p \leqslant \max\{|x|_p, |y|_p\}$ ($|\cdot|_p$ is here the p-adic norm), which gives rise to the wanted properties. (For more information about the origin of the p-adic physics and the corresponding historical overview see [10].)

One important topic in the non-Archimedean quantum theory is the investigation of spectral properties of a p-adic counterpart for Schrödinger operator, the so-called p-adic

Schrödinger-type operator $D^\alpha + V(x)$. The (fractional) differentiation operator over the p-adic numbers

$$(D^\alpha\varphi)(x) = \frac{1-p^\alpha}{1-p^{-\alpha-1}}\int_{\mathbb{Q}_p}|y|_p^{-\alpha-1}(\varphi(x-y)-\varphi(x))\,\mathrm{d}y$$

was first introduced in [11] and investigated in [9, 11]. (Here the integration is meant with respect to Haar measure on the additive group of field of p-adic numbers $\mathbb{Q}_p$. For the basic concepts from p-adic analysis used in this paper we refer the reader to [10].) The 'degree' α ($\alpha > 0$) of operator is an arbitrary positive real number. The value of α is inessential for further consideration: all results are valid for any $\alpha > 0$.

Especially, well investigated are (nonrandom) Schrödinger-type operators with radial potentials, i.e. operators of the form $D^\alpha + V(|x|_p)$ (see [2, 3, 9, 10]). It is, however, known from the real case that random operators (with random potentials) not only often give the more realistic description of real processes than nonrandom ones but also the study of their properties leads to the better understanding of the properties of nonrandom operators (for example, the study of spectral properties of nonrandom one-dimensional Schrödinger operators with power-decaying potentials was influenced by the study of Schrödinger operators with decaying random potentials, see [6–8] for the further information). So it is interesting to consider not only deterministic Schrödinger-type operators but also their random counterparts.

In the present paper we consider a Schrödinger-type operator with a random radial potential, i.e. with the radial potential that forms a random field. It means that $V(p^l)$ are random variables on a probability space (Ω, F, P) for all integers l. The first case under consideration in this paper is the p-adic analogue of the Anderson model, which was considered first by the author in [4]. In the Anderson model, we have

$$V(|x|_p) = \sum_{n=-\infty}^{+\infty} v_n(\omega)\delta(p^n - |x|_p)$$

where $v_n(\omega)$ are independent random variables. According to [5], chapter 9, we assume that $\Omega = \times_{n=-\infty}^{\infty}\mathbb{R}$, F is the σ-algebra generated by cylinder sets, P is the product measure $\prod_{n=-\infty}^{\infty} P_n$ (P_n is the distribution of v_n, i.e. $P_n(A) = P(v_n(\omega) \in A)$ for any Borel set A of real numbers) and the random variables are realized in such a way that $v_n(\omega) = \omega(n)$. We assume that v_n are distributed on $[a_n, b_n]$, where $a_n \in \mathbb{R}\bigcup\{-\infty\}$, $b_n \in \mathbb{R}\bigcup\{+\infty\}$ (i.e. $\mathrm{supp}(P_n) \subset [a_n, b_n]$)). We will consider the Anderson model in section 2. Then, in section 3 we consider another random model, namely, the model with random deviations.

The basic results from the nonrandom case which will be used in the present paper, were obtained by Kochubei in [2, 3]. We formulate the corresponding theorems here:

Theorem 1.1 *(due to [2, 3])*. *Let $p \neq 2$. Let $V(|x|_p)$ be locally bounded. We denote with H' the operator $(D^\alpha\varphi)(x) + V(|x|_p)\varphi(x)$ defined in $L_2(Q_p)$ with the domain $\mathcal{D}(Q_p)$ (the subspace of test functions, for the definition see [10]). Then the operator H' is closable and its closure ($H = \bar{H}'$) is self-adjoint. Moreover, H can be represented as the direct sum $H = H_1 \bigoplus H_2$, where H_1 and H_2 are both self-adjoint and H_2 has a complete system of eigenvectors. This representation is independent of the potential $V(|x|_p)$. The eigenvalues of H_2 are $p^{\alpha N} + V(p^{l-N})$ ($N \in \mathbb{Z}, l = 2, 3, \ldots$), each with finite multiplicity $(p-1)^2 p^{l-2}$ (if all eigenvalues are different, otherwise we have to add multiplicities).*

Theorem 1.2 *(due to [2, 3])*. *Let $p \neq 2$. Let the potential $V(|x|_p)$ satisfy the condition $V(p^l) \to 0$ for $l \to -\infty$. Then the essential spectrum of H_1 coincides with the set of all finite limit points of the sequence $V(p^l)$, $l \geqslant 0$.*

Let in the above-defined Anderson model the sequences $(a_n)_{n<0}$ and $(b_n)_{n<0}$ be bounded. Then it makes sense to consider the operator H_1^ω, because it follows from the boundedness of the sequences $(a_n)_{n<0}$ and $(b_n)_{n<0}$ that for almost all (with respect to P) ω from Ω theorem 1.1 is applicable and the operator $H^\omega = D^\alpha + V^\omega(|x|_p)$ has the decomposition $H^\omega = H_1^\omega \bigoplus H_2^\omega$, which is independent of the potential and therefore independent of ω. The following theorem was stated in [4]:

Theorem 1.3. *Let the random field $(v_n)_{n\in\mathbb{Z}}$ be as above, let v_n be identically distributed for $n \geqslant 0$ and let the sequences (a_n) and (b_n) satisfy the condition $a_n \to 0$ and $b_n \to 0$ for $n \to -\infty$. Then $\sigma_{ess}\left(H_1^\omega\right) = \operatorname{supp}(P_0)$ a.s. (almost sure) with respect to P.*

The proof of this theorem is based on the theorems 1.1 and 1.2, so it was proved in [4] only for the case $p \neq 2$, but theorem 1.3 is also true in the case $p = 2$, because theorems 1.1, 1.2 remain valid in the case $p = 2$ also (we have only to replace in the statement of theorem 1.1 the set $\{p^{\alpha N} + V(p^{l-N})\}_{N\in\mathbb{Z},l=2,3,\ldots}$ by the set $\{2^{\alpha N} + V(2^{1+l-N})\}_{N\in\mathbb{Z},l=2,3,\ldots}$) and the proof of theorem 1.3 is actually independent of the exact value of p. The ideas of the proofs of theorems 1.1 and 1.2 in the case $p = 2$ are quite similar to those in the case $p \neq 2$, so it is somewhat surprising that the corresponding proofs were not yet published somewhere. We sketch these proofs in section 4 (appendix) at the end of this paper for the aim of completeness.

2. Spectrum of the operator H_2 in Anderson model

Although the spectrum of the operator H_2^ω is completely known for each fixed ω (we have $\sigma(H_2^\omega) =$ the closure of the set $\{p^{\alpha N} + v_{l-N}(\omega)\}_{N\in\mathbb{Z},l\geqslant 2}$ in the case $p \neq 2$ and $\sigma(H_2^\omega) =$ the closure of the set $\{2^{\alpha N} + v_{1+l-N}(\omega)\}_{N\in\mathbb{Z},l\geqslant 2}$ in the case $p = 2$), the question about the location of the essential spectrum of H_2 is not so easy to answer. Generally, we can describe the essential spectrum of H_2 as the set of all limit points of the double-indexed sequence $\{p^{\alpha N} + v_{l-N}(\omega)\}_{N\in\mathbb{Z},l\geqslant 2}$ (if $p \neq 2$) or $\{2^{\alpha N} + v_{1+l-N}(\omega)\}_{N\in\mathbb{Z},l\geqslant 2}$ $(p = 2)$ which depends in a rather subtle way on the potential $V(\omega)$. The situation is much easier if we are only looking for the a.s. essential spectrum in the random Anderson model. This shows the following theorem:

Theorem 2.1. *Let $(a_n)_{n<0}$, $(b_n)_{n<0}$ in the Anderson model be bounded implying that $H_2(\omega)$ can be defined a.s. Let v_n be identically distributed for $n \geqslant 0$. Then $\sigma_{ess}\left(H_2^\omega\right)$ is with probability 1 equal to the closure of the set*

$$\bigcup_{n=-\infty}^{\infty} \{p^{\alpha n} + \operatorname{supp}(P_0)\}.$$

Proof. For fixed ω the essential spectrum of H_2^ω is the set of all finite limit points of the set $\{p^{\alpha N} + v_{l-N}(\omega)\}_{N\in\mathbb{Z},l\geqslant 2}$ in the case $p \neq 2$ or $\{2^{\alpha N} + v_{1+l-N}(\omega)\}_{N\in\mathbb{Z},l\geqslant 2}$ in the case $p = 2$. Note that (if $p \neq 2$) the point $p^{\alpha N_0} + v_{l_0-N_0}(\omega)$ lies in the $\sigma_{ess}(H_2)$ if and only if $p^{\alpha N_0} + v_{l_0-N_0}(\omega) = p^{\alpha N} + v_{l-N}(\omega)$ for infinite many different pairs (l, N) (in the case $p = 2$ we need to replace v_{l-N} by v_{l+1-N} and $v_{l_0-N_0}$ by $v_{1+l_0-N_0}$).

We will consider only the case $p \neq 2$. The case $p = 2$ can be treated identically taking into account the just-announced replacement.

Denote $B = \bigcup_{n=-\infty}^{\infty}\{p^{\alpha n} + \operatorname{supp}(P_0)\}$. We have to prove the following assertion: $P(A_0) = 0$, where

$$A_0 = \{\omega \in \Omega|\text{ the set of limit points of } (p^{\alpha n} + v_{l-n}(\omega)) \neq \bar{B}\}$$

($\bar{B}$ means the closure of B). We have $A_0 = A_1 \bigcup A_2$, where

$$A_1 = \{\omega \in \Omega | \exists \text{ a limit point of } (p^{\alpha n} + v_{l-n}(\omega)) \notin \bar{B}\}$$

and $A_2 = \{\omega \in \Omega | \exists x_0 \in \bar{B} : x_0 \text{ is not a limit point of } (p^{\alpha n} + v_{l-n}(\omega))\}$. So it is enough to prove $P(A_1) = P(A_2) = 0$. To do this we enumerate all finite subsets of $\mathbb{Z} \times \{\mathbb{N} \backslash \{1\}\}$ by B_k, $k = 1, 2, \ldots$ and all rational numbers by r_n, $n = 1, 2, \ldots$. We denote by M the set of pairs (m, n) of positive integers satisfying the condition

$$(r_n - 2^{-m}, r_n + 2^{-m}) \bigcap B \neq \emptyset \tag{1}$$

and denote $U_{m,n,N} = (r_n - 2^{-m} - p^{\alpha N}, r_n + 2^{-m} - p^{\alpha N})$.

Then

$$A_1 = \bigcup_{(m,n) \notin M} \tilde{A}_{m,n}$$

where

$$\tilde{A}_{m,n} = \{\omega | p^{\alpha N} + v^{\omega}_{l-N} \in (r_n - 2^{-m}, r_n + 2^{-m}) \text{ for infinitely many pairs } (N, l)\}.$$

We have for the set $\tilde{A}_{m,n}$ the following representation:

$$\tilde{A}_{m,n} = \left(\bigcap_{l=2}^{\infty} \bigcup_{j \geqslant l} \bigcup_{N \in \mathbb{Z}} \{v^{\omega}_{j-N} \in U_{m,n,N}\} \right) \bigcup \left(\bigcap_{N \in \mathbb{Z}} \bigcup_{|j| \geqslant |N|} \bigcup_{l=2}^{\infty} \{v^{\omega}_{l-j} \in U_{m,n,j}\} \right). \tag{2}$$

From $(m, n) \notin M$ follows $U_{m,n,N_1} \bigcap \text{supp}(P_0) = \emptyset$ for every N_1 from $\mathbb{Z}$, so we have $P\{v_{l_1} \in U_{m,n,N_1}\} = 0$ for each pair (l_1, N_1). This implies, taking into account (2), that $P(\tilde{A}_{m,n}) = 0$. So we have $P(A_1) = 0$.

For A_2 we have the representation

$$A_2 = \bigcup_{k=1}^{\infty} \bigcup_{(m,n) \in M} A_{k,m,n}$$

where

$$A_{k,m,n} = \{\omega | p^{\alpha N} + v^{\omega}_{l-N} \notin (r_n - 2^{-m}, r_n + 2^{-m}) \text{ if and only if } (N, l) \notin B_k\}.$$

We now prove that for each $k \in \mathbb{N}$ and (m, n) from M holds $P(A_{k,m,n}) = 0$. Then $P(A_2) = 0$ will follow from the σ-additivity of the probability. Let k and (m, n) be fixed. $(m, n) \in M$ implies (1), so some integer N_0 exists, such that

$$(r_n - 2^{-m}, r_n + 2^{-m}) \bigcap \{p^{\alpha N_0} + \text{ supp}(P_0)\} \neq \emptyset.$$

Denote by B_{k,N_0} the set $\{l : (N_0, l) \in B_k\}$. This set is evidently finite. We can represent $A_{k,m,n}$ in the following form:

$$A_{k,m,n} = \bigcap_{(N,l) \notin B_k} \{v^{\omega}_{l-N} \notin U_{m,n,N}\} \bigcap_{(N,l) \in B_k} \{v^{\omega}_{l-N} \in U_{m,n,N}\}.$$

So the set $A_{k,m,n}$ is measurable. We have the inclusion

$$A_{k,m,n} \subset \bigcap_{l \notin B_{k,N_0}} \{v^{\omega}_{l-N_0} \notin U_{m,n,N_0}\}$$

from which follows

$$P(A_{k,m,n}) \leqslant \prod_{l \notin B_{k,N_0}} P_{l-N_0}(\mathbb{R} \backslash U_{m,n,N_0}) = 0$$

since $P_{l-N_0} = P_0$ for each $l \geqslant N_0$ and since $U_{m,n,N_0} \bigcap \operatorname{supp}(P_0) \neq \emptyset$ and therefore $P_0(\mathbb{R} \setminus U_{m,n,N_0}) = 1 - P_0(U_{m,n,N_0}) < 1$. □

3. The model with random deviations

In the present section we consider Schrödinger-type operators with radially symmetric random potentials $V(|x|_p)$ of the following form:

$$V(|x|_p) = \sum_{n=-\infty}^{+\infty} v_n \delta(p^{n+\xi_n(\omega)} - |x|_p) \tag{3}$$

where $(v_n)_{n\in\mathbb{Z}}$ is a sequence of real numbers satisfying the condition $v_n = v_0$ for all $n \geqslant 0$ (so $(v_n)_{n\geqslant 0}$ is a constant sequence) and $\xi_n(\omega)$ are random variables on a probability space (Ω, F, P) for all integers n. We assume (ξ_n) to be a sequence of independent identically distributed discrete random variables with integer values such that M from $\mathbb{Z}$ exists for which holds $\xi_n \geqslant M$ a.s. for all n. Denote $q_k = P(\xi_n = k)$ for $k \in \mathbb{Z}$ (q_k is independent of n by assumption) and let μ_ξ be the measure $\sum_{k=M}^{+\infty} q_k \delta(x-k)$. Then μ_ξ is the distribution of ξ_n for any n (so $\mu_\xi(A) = P(\xi_n(\omega) \in A)$ for any Borel set A of real numbers).

We can assume without loss of generality that (see [1], chapter 9, or [5], chapter 1) $(\Omega, F, P) = \left(\times_{n=-\infty}^{\infty}\mathbb{R}, F, \mu\right)$, where F is the σ-algebra generated by cylinder subsets of Ω, μ is the product measure $\prod_{n=-\infty}^{\infty} \mu_\xi$ and the random variables are realized in such a way that $\xi_n(\omega) = \omega(n)$.

We introduce new random variables η_k, $k \in \mathbb{Z}$, by the formula

$$\eta_k = \sum_{n=-\infty}^{+\infty} \delta(n + \xi_n(\omega) - k).$$

Lemma 3.1. *η_k are identically distributed.*

Proof. We enumerate all finite subsets of $\mathbb{Z}$ by G_l, $l = 1, 2, \ldots$ Then we have the equality

$$P\{\eta_k = n_0\} = \sum_{G_l : |G_l| = n_0} \prod_{j\in G_l} P(\xi_j = k - j) \prod_{j \notin G_l} (1 - P(\xi_j = k - j))$$

which shows that $P\{\eta_k = n_0\}$ is actually independent of k for each n_0 ($|G_l|$ is the number of elements in G_l). □

Lemma 3.2. *Let $N_\xi = \operatorname{card}\{q_k > 0\}$ satisfy $N_\xi > 1$, i.e. $N_\xi \in \{2, 3, \ldots, \infty\}$ (in the case $N_\xi = 1$ the potential is actually nonrandom since $\xi_n = a$ a.s. for some a for all n, so this exclusion is quite natural). Then*

$$P\left(\bigcap_{n=1}^{\infty}\bigcup_{j=n}^{\infty}\{\omega : \eta_j(\omega) = m\}\right) = 1 \qquad \textit{for any} \quad m = 0, 1, \ldots, N_\xi$$

and

$$P(\{\omega : \eta_j(\omega) = m\}) = 0 \qquad \textit{for any } j \textit{ and for any } m \neq 0, 1, 2, \ldots, N_\xi.$$

Proof. Let T_+ and T_- be the right and left shifts on the space Ω, i.e. $T_\pm\omega(n) = \omega(n \pm 1)$ and let m be from $\{0, 1, \ldots, N_\xi\}$. We now show that the events $\bigcap_{n=1}^{\infty}\bigcup_{j=n}^{\infty}\{\omega : \eta_j(\omega) = m\}$ are

$T_\pm$-invariant. We have

$$T_\pm(\{\omega : \eta_j(\omega) = m\}) = T_\pm\left(\left\{\omega : \sum_{n=-\infty}^{+\infty} \delta(n+\omega(n)-j) = m\right\}\right)$$
$$= \left\{\omega : \sum_{n=-\infty}^{+\infty} \delta(n+\omega(n\pm 1)-j) = m\right\}$$
$$= \left\{\omega : \sum_{n=-\infty}^{+\infty} \delta(n+\omega(n)-j\mp 1) = m\right\} = \{\omega : \eta_{j\mp 1}(\omega) = m\} \quad (4)$$

implying

$$T_\pm\left(\bigcup_{j=n}^{\infty}\{\omega : \eta_j(\omega) = m\}\right) = \bigcup_{j=n\mp 1}^{\infty}\{\omega : \eta_j(\omega) = m\}$$

and

$$T_\pm\left(\bigcap_{n=1}^{\infty}\bigcup_{j=n}^{\infty}\{\omega : \eta_j(\omega) = m\}\right) = \bigcap_{n=1}^{\infty}\bigcup_{j=n}^{\infty}\{\omega : \eta_j(\omega) = m\}.$$

The shifts $T_\pm$ are the metrically transitive automorphisms of the probability space $\Omega = \mathbb{R}^{\mathbb{Z}}$ (see [5], example 1.14), i.e. each $T_\pm$-invariant event has probability 0 or 1, so it follows that

$$P\left(\bigcap_{n=1}^{\infty}\bigcup_{j=n}^{\infty}\{\omega : \eta_j(\omega) = m\}\right) \in \{0, 1\}.$$

So, it will be sufficient now to show that $P\left(\bigcap_{n=1}^{\infty}\bigcup_{j=n}^{\infty}\{\omega : \eta_j(\omega) = m\}\right) > 0$. From the continuity of probability follows

$$P\left(\bigcap_{n=1}^{\infty}\bigcup_{j=n}^{\infty}\{\eta_j(\omega) = m\}\right) = \lim_n P\left(\bigcup_{j=n}^{\infty}\{\eta_j(\omega) = m\}\right) \geqslant \lim_n P(\{\eta_n(\omega) = m\}).$$

Since η_n are identically distributed we can consider one fixed n_0. We have to prove $P\left(\left\{\eta_{n_0}(\omega) = m\right\}\right) > 0$ for each $m = 0, \ldots, N_\xi$. We assume first $m > 0$. Let us choose $n_i, i = 1, \ldots, m$ so that $q_{n_0-n_i} > 0$ (we can do it since $m \leqslant N_\xi$). Then we have

$$\bigcap_{i=1}^{m}\left\{\omega : \xi_{n_i} = n_0 - n_i\right\}\bigcap\bigcap_{j\neq n_i}\left\{\omega : \xi_j \neq n_0 - j\right\} \subset \left\{\eta_{n_0}(\omega) = m\right\}$$

implying

$$P\left\{\eta_{n_0}(\omega) = m\right\} \geqslant \prod_{i=1}^{m} P\left\{\omega : \xi_{n_i} = n_0 - n_i\right\}\prod_{j\neq n_i} P\{\omega : \xi_j \neq n_0 - j\}$$
$$= \prod_{i=1}^{m} q_{n_0-n_i}\prod_{j\neq n_i}\left(1 - q_{n_0-j}\right) > 0$$

where we have used the independence of η_k and the property $\sum q_i = 1$, implying $\prod(1-q_i) > 0$ ($q_i \neq 1$ for all i since $N_\xi > 1$).

For $m = 0$ we have

$$P\{\eta_{n_0}(\omega) = 0\} \geqslant \prod_{j=-\infty}^{\infty} P\{\omega : \xi_j \neq n_0 - j\} \prod_{j=-\infty}^{\infty} (1 - q_{n_0-j}) > 0$$

by the same reason as in the case $m > 0$. The second statement of the present lemma is trivial. □

Theorem 3.3.

(i) Let the potential (3) satisfy the conditions $N_\xi < \infty$ and $v_n \to 0, n \to -\infty$ (N_ξ is as in lemma 3.2 equal to card$\{q_k > 0\}$*). Then the operator $H^\omega = D^\alpha + V^\omega$ has with probability 1 the decomposition $H^\omega = H_1^\omega \bigoplus H_2^\omega$, which is independent of ω, and we have with probability 1*

$$\sigma_{ess}\left(H_1^\omega\right) = \{0, v_0, 2v_0, 3v_0, \ldots, N_\xi v_0\}$$

and

$$\sigma_{ess}\left(H_2^\omega\right) = \{0, v_0, 2v_0, \ldots, N_\xi v_0\} \bigcup \bigcup_{m=-\infty}^{\infty} (p^{\alpha m} + \{0, v_0, 2v_0, \ldots, N_\xi v_0\}). \tag{5}$$

(ii) The decomposition $H^\omega = H_1^\omega \bigoplus H_2^\omega$ and the formula for the essential spectrum $\sigma_{ess}\left(H_1^\omega\right)$

$$\sigma_{ess}\left(H_1^\omega\right) = \{0, v_0, 2v_0, 3v_0, \ldots, N_\xi v_0\} \qquad a.s.$$

remain valid if we replace the conditions $N_\xi < \infty$ and $v_n \to 0, n \to -\infty$ by the condition $v_n = 0$ for $n < M_2$ and $v_n = v_0$ otherwise, where M_2 is some number (so N_ξ can in this case be equal to ∞, in which case we have $\sigma_{ess}(H_1) = \{0, v_0, 2v_0, 3v_0, \ldots\}$).

Proof.

(i) From $N_\xi < \infty$ it follows that $M_1 \in \mathbb{Z}$ exists, such that $\xi_n \leqslant M_1$ a.s., implying $\delta(p^{n+\xi_n} - p^l) = 0$ for all $n < l - M_1$. On the other hand, $\xi_n \geqslant M$ implies $\delta(p^{n+\xi_n} - p^l) = 0$ for all $n > l - M$. So we have a.s. the equality

$$V(p^l) = \sum_{n=l-M_1}^{l-M} v_n \delta(p^{n+\xi_n(\omega)} - p^l) \tag{6}$$

from which it follows that V is a.s. locally bounded. Therefore, we can apply theorem 1.1 to obtain the announced decomposition.

From theorem 1.2 we know that $\sigma_{ess}(H_1)$ is equal to the set of all limit points of the sequence $(V(p^l))_{l>0}$. Equation (6) implies that

$$V(p^l) = v_0 \sum_{n=l-M}^{l-M_1} \delta(p^{n+\xi_n(\omega)} - p^l)$$

for all $l > M$. Then the statement of the present theorem concerning $\sigma_{ess}(H_1)$ follows from lemma 3.2.

It was already discussed in section 2 that $\sigma_{ess}(H_2)$ is equal to the set of all limit points of the double-indexed sequence $(p^{\alpha m} + V(p^{l-m}))_{m\in\mathbb{Z},l\geqslant 2}$ if $p \neq 2$ and $2^{\alpha m} + V(2^{1+l-m}))_{m\in\mathbb{Z},l\geqslant 2}$ for $p = 2$. To obtain (5) we note that (6) implies that $V(p^l)$ is (uniform in ω) bounded and that all limit points we are looking for are therefore obtained by the limiting process $m \to -\infty$ by fixed l or by the limiting process $l \to \infty$ by fixed m. Now it is sufficient to apply lemma 3.2. (ii) It is clear that we have only to consider the case $N_\xi = \infty$. In this case we have instead of (6) the formula

$$V(p^l) = \sum_{n=M_2}^{l-M} v_n \delta(p^{n+\xi_n(\omega)} - p^l) = v_0 \sum_{n=M_2}^{l-M} \delta(p^{n+\xi_n(\omega)} - p^l) \tag{7}$$

implying $V(p^l) \to 0, l \to -\infty$. We apply lemma 3.2 to obtain the statement about $\sigma_{ess}(H_1)$ as in (i). □

Appendix. 2-adic nonrandom Schrödinger-type operator

The case $p = 2$ often requires the exceptional consideration in the p-adic analysis. This is also the case in the study of p-adic Schrödinger-type operators. The reason for this is that the eigenfunctions of the operator D^α have somewhat different properties for $p \neq 2$ and $p = 2$. In spite of these differences we can use the ideas from [2] to analyse the spectral properties of Schrödinger-type operator $D^\alpha + V(|x|_p)$.

We recall that the operator D^α is for each p self-adjoint on $L_2(\mathbb{Q}_2)$ and has a complete system of eigenfunctions [10]. In the case $p = 2$ the set of eigenvalues of D^α is the sequence $(2^{\alpha N})_{N\in\mathbb{Z}}$ and to each eigenvalue $2^{\alpha N}$ corresponds the following system of eigenfunctions (they are sometimes called Vladimirov functions):

$$\psi^l_{N,k,\varepsilon} = 2^{\frac{N-1}{2}} \delta(|x|_2 - 2^{1+l-N}) \chi_2(\varepsilon 2^{l-2N} x^2 + 2^{l-N-k} x)$$

for $l = 2, 3, \ldots, k = 0, 1$ and $\varepsilon = 1 + \varepsilon_1 2 + \cdots + \varepsilon_{l-2} 2^{l-2}$ $(\varepsilon_i \in \{0, 1\})$ and

$$\psi^1_{N,k,0} = 2^{\frac{N-1}{2}} [\omega(2^N |x - k2^{N-2}|_2) - \delta(|x - k2^{N-2}|_2 - 2^{1-N})]$$

for $k = 0, 1$, where $\chi_2(\cdot)$ is a canonical additive character of $\mathbb{Q}_2 (\chi_2(x) = \mathrm{e}^{2\pi \mathrm{i}\{x\}_2}$, where $\{x\}_2$ is a fractional part of x, see [10]) and ω and δ are defined by

$$\delta(a) = \begin{cases} 1 & a = 0 \\ 0 & a \neq 0 \end{cases} \qquad \omega(a) = \begin{cases} 1 & a \leqslant 1 \\ 0 & a > 1. \end{cases}$$

It is clear that for every $l \geqslant 2$ and every N, k and ϵ holds

$$\mathrm{supp}\big(\psi^l_{N,k,\varepsilon}\big) \subset \{x \in \mathbb{Q}_2 : |x|_2 = 2^{1+l-N}\}. \tag{8}$$

In the case $l = 1$ we need the more detailed information on the eigenfunctions. For the function $\psi^1_{N,0,0}$ we have the representation

$$\psi^1_{N,0,0} = 2^{\frac{N-1}{2}} \begin{cases} 0 & |x|_2 > 2^{1-N} \\ -1 & |x|_2 = 2^{1-N} \\ 1 & |x|_2 < 2^{1-N}. \end{cases} \tag{9}$$

For the functions $\psi^1_{N,1,0}$ we have a somewhat more complicated formula:

$$\psi^1_{N,1,0} = 2^{\frac{N-1}{2}} \begin{cases} 0 & |x|_2 > 2^{2-N} \\ -1 & |x|_2 = 2^{2-N} \quad \text{and} \quad |x - 2^{N-2}|_2 = 2^{1-N} \\ 1 & |x|_2 = 2^{2-N} \quad \text{and} \quad |x - 2^{N-2}|_2 < 2^{1-N} \\ 0 & |x|_2 < 2^{2-N} \end{cases}.$$

Particularly holds for all N:

$$\mathrm{supp}\big(\psi^1_{N,1,0}\big) = \{x \in \mathbb{Q}_2 : |x|_2 = 2^{2-N}\} \quad \text{and} \quad \mathrm{supp}\big(\psi^1_{N,0,0}\big) = \{x \in \mathbb{Q}_2 : |x|_2 \leqslant 2^{1-N}\}. \tag{10}$$

Schrödinger-type operator $D^\alpha + V(|x|_p)$ with locally bounded $V(|x|_p)$ is initially defined on the subspace $\mathcal{D}(\mathbb{Q}_p)$ consisting of all locally constant functions with compact supports (see [10]). As usual in mathematical physics the question of particular interest is the self-adjoint realization of this operator. We will see that such realization is given exactly as in the case $p \neq 2$ by the operator H defined by

$$H\phi(x) = D^\alpha \phi(x) + V(|x|_p)\phi(x) \qquad \text{on the subspace}$$
$$\{\phi \in L_2(\mathbb{Q}_2) : D^\alpha \phi(x) + V(|x|_p)\phi(x) \in L_2(\mathbb{Q}_2)\}. \tag{11}$$

We denote by $\mathcal{H}_1$ and $\mathcal{H}_2$ the closed subspaces of $L_2(\mathbb{Q}_2)$ spanned by functions $\psi^l_{N,k,\varepsilon}$ with $l = 1$ (in which case holds $\varepsilon = 0$) and $l \geqslant 2$ respectively. The following lemma is the 2-adic analogue of the result from [2]:

Lemma 4.1. *The subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ reduce the operator H.*

Proof. The proof is essentially the same as in [2]. The only properties of the eigenfunctions $\psi^l_{N,k,\varepsilon}$ which have to be used are that their supports lie in the sets of the form $\{x \in \mathbb{Q}_2 : |x|_2 = 2^m\}$ for $l \geqslant 2$ and $\{x \in \mathbb{Q}_2 : |x|_2 \leqslant 2^m\}$ for $l = 1$. □

Let H_j be the part of the operator H on the subspace $\mathcal{H}_j$, $j = 1, 2$. The operator H_2 has the especially simple structure: H_2 is self-adjoint and possesses a complete system of eigenfunctions. (Particularly we have $\sigma(H_2) = \sigma_{ess}(H_2) = \overline{\{2^{\alpha N} + V(2^{1+l-N})\}_{N\in\mathbb{Z},l\geqslant 2}}$.) In contrast to the p-adic case with $p \neq 2$ we can proceed to represent the operator H_1 as the orthogonal sum. We denote by $\mathcal{H}_{1,0}$ and $\mathcal{H}_{1,1}$ the closed subspaces of $L_2(\mathbb{Q}_2)$ spanned by functions $\psi^1_{N,k,0}$ with $k = 0$ and $k = 1$, respectively.

Lemma 4.2. *The subspaces $\mathcal{H}_{1,0}$ and $\mathcal{H}_{1,1}$ reduce the operator H_1.*

Proof. The proof is exactly the same as that of the previous lemma. □

Let $H_{1,j}$ be the part of the operator H_1 on the subspace $\mathcal{H}_{1,j}$, $j = 0, 1$. The operator $H_{1,1}$ is self-adjoint and has a complete system of eigenfunctions in $\mathcal{H}_{1,1}$. All eigenvalues of $\mathcal{H}_{1,1}$ (which are $2^{\alpha N} + V(2^{2-N})$) are simple, so we have

$$\sigma(H_{1,1}) = \overline{\{2^{\alpha N} + V(2^{2-N})\}_{N\in\mathbb{Z}}}$$

whereas the essential spectrum $\sigma_{ess}(H_{1,1})$ consists of all accumulation points of the sequence $(V(2^k))_{k>0}$. Let us denote by $\mathcal{H}_{1,0,-}$ and $\mathcal{H}_{1,0,+}$ the closed subspaces of $L_2(\mathbb{Q}_2)$ spanned by functions $\psi^1_{N,0,0}$ with $N \leqslant \nu$ and $N > \nu$ respectively (ν is some integer). We define the function V_ν by $V_\nu(|x|_2) = V(|x|_2)\omega(2^{-\nu}|x|_2)$. Thus we have $V_\nu(|x|_2) = 0$ for every x with $|x|_2 > 2^\nu$ and $V_\nu(|x|_2) = V(|x|_2)$ in the opposite case. Therefore holds

$$(H_{1,0} - V_\nu)f(x) = \begin{cases} D^\alpha f(x) & |x|_2 \leqslant 2^\nu \\ (D^\alpha + V(|x|_2))f(x) & |x|_2 > 2^\nu \end{cases}$$

from which follows $(H_{1,0} - V_\nu)|_{H_{1,0,+}} = D^\alpha|_{H_{1,0,+}}$.

Lemma 4.3. *The subspaces $\mathcal{H}_{1,0,-}$ and $\mathcal{H}_{1,0,+}$ reduce the operator $H_{1,0} - V_\nu$.*

Proof. The proof is similar to those of two previous lemmas. See [2, 3] for more details. □

Theorem 4.4 *(=theorem 1.1 for $p = 2$)*. *Defined by (11) operator H is self-adjoint and has the decomposition*

$$H = \left(V_\nu + \left(D^\alpha|_{H_{1,0,-}} + W_\nu\right) \oplus D^\alpha|_{H_{1,0,+}}\right) \oplus H_{1,1} \oplus H_2$$

where $W_\nu(|x|_2) = V(|x|_2) - V_\nu(|x|_2)$.

Proof. The proof follows from three previous lemmas (see also [2]). □

Lemma 4.5. *The essential spectrum of the operator* W_ν *on* $\mathcal{H}_{1,0,-}$ *coincides with the set of all finite accumulation points of the sequence* $\{V(2^l)\}_{l>\nu}$.

Proof. It is clear that the spectrum of W_ν is contained in the closure of the set $0 \cup \{V(2^l)\}_{l>\nu}$. Actually, each element of this set is an eigenvalue, because from (9) it follows that for each $N \leqslant -\nu$ the function

$$\phi^{(N)} = 2^{\frac{1-N}{2}} \psi^1_{N,0,0} + \sum_{n=N+1}^{-\nu} 2^{n-N-1} 2^{\frac{1-n}{2}} \psi^1_{n,0,0}$$

has the property supp $\phi^{(N)} \subset \{x : |x|_2 \leqslant 2^\nu\} \cup \{x : |x|_2 = 2^{1-N}\}$, so $\phi^{(N)}$ is the eigenfunction corresponding to the eigenvalue $V(2^{1-N})$.

It remains now only to prove that those eigenvalues, which appear in the sequence $\{V(2^l)\}_{l>\nu}$ a finite number of times, have finite multiplicities. Let $\phi \in \mathcal{H}_{1,0,-}$ be an eigenfunction of W_ν corresponding to one such eigenvalue $V(2^{l_0})$. Let us consider Fourier series expansion for $\phi : \phi = \sum_{N=-\infty}^{-\nu} C_N \psi^1_{N,0,0}$. Then from $W_\nu \phi = V(2^{l_0})\phi$ follows now

$$\sum_{N=-\infty}^{-\nu} C_N W_\nu \psi^1_{N,0,0} = \sum_{N=-\infty}^{-\nu} V(2^{l_0}) C_N \psi^1_{N,0,0}.$$

We have for x from the set $\{x : |x|_2 = 2^m\}(m > \nu) : \psi^1_{N,0,0} = 2^{\frac{N-1}{2}}$ if $-N \geqslant m$, $\psi^1_{N,0,0} = -2^{\frac{N-1}{2}}$ if $-N + 1 = m$ and $\psi^1_{N,0,0} = 0$ if $-N \leqslant m - 1$. So we have for all x with $\{x : |x|_2 = 2^m\}$ the equality

$$\sum_{N=-\infty}^{-m} C_N (V(2^m) - V(2^{l_0})) 2^{\frac{N-1}{2}} = (V(2^m) - V(2^{l_0})) C_{1-m} 2^{\frac{-m}{2}}. \tag{12}$$

We can add $(V(2^m) - V(2^{l_0})) C_{1-m} 2^{\frac{-m}{2}}$ to both sides to obtain

$$\sum_{N=-\infty}^{-m+1} C_N (V(2^m) - V(2^{l_0})) 2^{\frac{N-1}{2}} = 2(V(2^m) - V(2^{l_0})) C_{1-m} 2^{\frac{-m}{2}}.$$

On the other hand, the left-hand side must be equal to $(V(2^m) - V(2^{l_0})) C_{2-m} 2^{\frac{-m+1}{2}}$ according to the formula (12), so we have $2C_{1-m} 2^{\frac{-m}{2}} = C_{2-m} 2^{\frac{-m+1}{2}}$ for all m such that $V(2^m) \neq V(2^{l_0})$. Since the set of all m such that $V(2^m) = V(2^{l_0})$ is finite by assumption, we can conclude that all the coefficients C_N with $N < N_1 = 1 - \max\{m : V(2^m) = V(2^{l_0})\}$ are defined by the value of C_{N_1}. So the space of eigenfunctions corresponding to $V(2^{l_0})$ is finite-dimensional. □

Theorem 1.2 in the case $p = 2$ follows now from the last lemma and from theorem 4.4 exactly as it is in the case $p \neq 2$ (see [2, 3]).

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